Functions and Graphs

Some graphs are difficult to draw. You may use some software to help you.

e.g. Winplot (http://math.exeter.edu/rparris/) , GRAPES (http://www.criced.tsukuba.ac.jp/grapes/)

1. Sketch the graphs of :

(a) $y = |x|$ **(b)** $|y| = |x|$ **(c)** $|x| + |y| = 2$

(d) $|x + y| = 2$ **(e)** $y = |x + 2| + |x + 5|$ **(f)** $y = |x^2 - 3x + 2|$

2. [x] denotes the greatest integer not greater than x. Sketch the graphs of :

(a) $y = [x]$ **(b)** $y = x - [x]$ **(c)** $y = \sqrt{x - [x]}$ **(d)** $y = [x] + \sqrt{x - [x]}$

3. **(a)** Show that $f(x) + f(-x)$ is an even function and that $f(x) - f(-x)$ is an odd function.

(b) Show that every function $f(x)$ can be expressed as the sum of an even function and an odd function.

4. **(a)** Show that $\sin^2 x$ is a periodic function and find its least period.

(b) Is $\sin (x^2)$ periodic ?

(c) Find the least period of : **(i)** $2 \sin 3x + 3 \sin 2x$; **(ii)** $|\sin x| + |\cos x|$.

5. The function f has the following properties :

(a) $f(x) = 4x$ for $0 \le x \le 2/3$;

(b) $f(x) = 8 (1 - x)$ for $2/3 \le x \le 1$;

(c) $f(x)$ is an odd function ;

(d) $f(x)$ is periodic with period 2.

Sketch the graph of $f(x)$ for $-3 \le x \le 3$.

6. **(a)** The functions $f(x)$ and $g(x)$ are real for real values of x and satisfy the equations

$f(x + y) = f(x) g(y) + g(x) f(y)$; $g(x + y) = g(x) g(y) - f(x) f(y)$

for all values of x and y . Given that $f(x)$ is not identically zero, prove that

(i) $f(0) = 0$ **(ii)** $g(0) = 1$

(b) If further $f'(0) = 1$, $g'(0) = 0$, prove that

(i) $f'(x) = g(x)$ **(ii)** $g'(x) = - f(x)$

(c) Determine the functions $f(x)$ and $g(x)$.

7. Show that if f , g are even functions, then so is f + g .

Show that the polynomial $\sum_{i=0}^{n} a_i x^{2i}$ is an even function.

8. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. f is said to be even if $f(-x) = f(x)$. f is odd if $f(-x) = -f(x)$.
Decide whether the following statements are true or false. Given proofs for true statements and counter examples for false statements.

(a) f is odd $\Rightarrow |f|$ is odd ;

(b) $\frac{1}{2}[g(x)+g(-x)]$ is even for every g ;

(c) if f is injective and the domain of f $\neq$ {0} , then f is not even ;

(d) if f is increasing, then f is not even ;

(e) if f , g are increasing, then so are f + g , fg , f **o** g .

9. Sketch the curves of

(a) $y = x^3 + x$ **(b)** $y = \frac{6x}{x^2+3}$ **(c)** $y = \frac{x-1}{x^2}$ **(d)** $y = x + 3 + \frac{4}{x-1}$

(e) $x^3 - y^3 = 2^3$ **(f)** $y = \frac{x^3}{x^2-1}$

10. Find the maximum, minimum and point of inflexion value(s) of $y = (2x-1)^{1/3}(x-1)^{2/3}$.

11. Sketch the curve $y = e^{-x^2}$. Perpendiculars are drawn from a variable point on the curve to the coordinate axes. Find the maximum area enclosed by these perpendiculars and the axes.

Show also that the minimum distance of a point on the curve from the origin is $\sqrt{\frac{\ln 2+1}{2}}$.

12. If $y = e^{-bt\cot\alpha}\sin bt$, where $t \geq 0$, b and α are constants and $0 < \alpha < \pi/2$, show that

$$\frac{dy}{dt} = b\csc\alpha\ e^{-bt\cot\alpha}\sin(bt+\pi-\alpha)$$ and find the value of $\frac{d^2y}{dt^2}$ in the same form .

Find the values of t which make y **(i)** a maximum **(ii)** a minimum,
and show that the largest maximum is $e^{-\alpha\cot\alpha}\sin\alpha$.

Sketch the graph of y when $b = \frac{\pi}{2}$ and $\alpha = \frac{\pi}{4}$, showing the graphs of $y = \pm e^{-\frac{\pi}{2}t}$ on the same axes.

13. Assuming that $a, b \in$ **+R** , establish the following properties of the curve $y = ax - \sin bx$,

(a) The gradient is always between $a - b$ and $a + b$.

(b) There are points of inflexion which lie on $y = ax$.

(c) There are maxima and minima if $a < b$;

the maximum points lie on $y = ax + \sqrt{1-\frac{a^2}{b^2}}$ and the minimum points lie on $y = ax - \sqrt{1-\frac{a^2}{b^2}}$.

(d) Illustrate these properties by sketches of the curve between $x = 0$ and $x = \frac{4\pi}{b}$ when

(i) $a > b$, **(ii)** $a < b$.

14. Show that the curve whose equation is $y^2 = \frac{a^2(x-a)}{x}$, where $a > 0$, has the lines $y = \pm a$, $x = 0$ as asymptotes. Sketch the graph. Find the equations of the tangents which pass through the origin and deduce, or prove otherwise, that the equation in x , $\lambda x^3 = a^2(x-a)$, $\lambda \geq 0$, has three real roots (including double roots) if $0 < \lambda \leq 4/27$, but only one real root if λ does not lie between these limits.

15. Find the equation of the tangent at the point t on the curve defined by the parametric equations $x = 3t^2$, $y = 2t^3$. Show that, in general, three tangents to the curve pass through a given point (x_1, y_1).

(i) Show that $x_1 > 0$ is a necessary condition for the three tangents to be real.

(ii) Find sufficient conditions for the three tangents to be real and distinct.

16. Show that if the tangent at the point (x_0, y_0) on the curve $x = a\cos^2 t \sin t$, $y = a \cos t \sin^2 t$ meets the x-axis at the point $(x_1, 0)$, then x_1 does not lie between 0 and .

Show also that the loop of the curve corresponding to the range $0 < t < \frac{\pi}{2}$ has an area $\frac{\pi a^2}{32}$.

17. Show that the function $\frac{e^{ax}}{1+x^2}$ has a maximum or a minimum value if $|a| < 1$, but that there are no turning points if $|a| > 1$. Draw rough graphs of the function for the cases $a = \frac{1}{2}$, $a = 1$, $a = 2$, for values of x from $-\infty$ to $+\infty$, showing clearly how they differ.

18. **(i)** Find the values of a and b if the curve $y = \frac{e^{2x}}{a+bx}$ has a stationary value $y = \frac{1}{2}$ when $x = 0$. Determine whether the stationary value is a maximum or minimum by sketching the graph, or otherwise.

(ii) Find the equation of the tangent from the origin to the curve $y = e^{ax}$. For what values of k does the equation $kx = e^{ax}$ have no real roots ?

19. Let $f(x) = \frac{(x-1)^3}{(x+1)^2}$ be a real-valued function defined on the real line except $x = -1$.

(a) Prove that the straight lines $x = -1$ and $y = x - 5$ are asymptotes of the graph $y = f(x)$.

(b) Find, if any

(i) the local maxima and minima of $f(x)$;

(ii) the intersections of the graph $y = f(x)$ and its asymptotes.

(c) Sketch the graph of $y = f(x)$.

(d) Sketch the graph of $y = |f(x)|$.

20. The function f is defined for all real numbers x. For each of the following statements give proof or counter-example as appropriate.

(i) If $f(x) \ge f(y)$ whenever $x > y$, then $f'(x) \ge 0$ for all x.

(ii) If $f(x) > f(y)$ whenever $x > y$, then $f'(x) > 0$ for all x.

(iii) If $f'(x) > 0$ for all x, then $f(x) > f(y)$ whenever $x > y$.

(iv) If $f'(x) \ge 0$ for all x, then $f(x) \ge f(y)$ whenever $x > y$.